Stochastic lattice models with several absorbing states

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We study two models with n equivalent absorbing states that generalize the Domany-Kinzel cellular automaton and the contact process. Numerical investigations show that for $n=2$ both models belong to the same universality class as branching annihilating walks with an even number of offspring. Unlike previously known models, these models have no explicit parity conservation law. [S1063-651X(97)07401-1]

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I. INTRODUCTION

The study of stochastic lattice models exhibiting a continuous phase transition from an active phase into an absorbing state is a field of growing interest. In these models the dynamical processes take place close to an absorbing state, i.e., a configuration once reached, the system cannot escape from. Most of them belong to the universality class of directed percolation (DP); the best known examples are DP lattice models [1], the Domany-Kinzel cellular automaton [2], the contact process [3], Schlögl's first and second model [4], and branching annihilating walks with an odd number of offsprings [5]. In a field-theoretical formulation these models can be related to Reggeon field theory [6], which was proven to be in the same universality class as directed percolation [7].

The variety of DP models led Janssen and Grassberger to the conjecture that in one-component models all continuous phase transitions from an active phase to a *single* absorbing state are in the DP universality class [8]. However, the known examples for DP include even more complicated systems, e.g., multicomponent systems [9] and models with several absorbing states [10]. Some models that were initially thought to be in different classes were later found to belong to the DP class as well [11]. Thus the directed percolation universality class is extremely robust and covers a wide range of models.

Among the models with absorbing states only a few exceptions are known that do not belong to the DP universality class. During the past few years it became clear that they represent a universality class that is different from that of directed percolation. The known examples are the models A and B of probabilistic cellular automata [12,13], nonequilibrium kinetic Ising models with combined zero- and infinite-temperature dynamics [14], interacting monomer-dimer models [15], and branching annihilating walks (BAW's) with an even number of offspring [16–20]. The common feature of all these models is that the number of particles (or kinks) is conserved mod2. Therefore the class is sometimes referred to as the *parity-conserving* (PC) class.

Initially parity conservation was thought to be the reason for the emergence of the different universality class. However, Park and Park [21] recently showed that in the example of an interacting monomer-dimer model a weak parity-conserving external field can force the system back to the DP class. They concluded that the essential property of the class is not parity conservation, but a symmetry among different absorbing states.

In order to address this question, we propose two one-dimensional models that exhibit a phase transition from an active to an inactive phase consisting of n equivalent absorbing states. These models generalize two well-known stochastic models for directed percolation, namely, the Domany-Kinzel cellular automaton [2] and the contact process [3]. We can conclude from numerical simulations that for $n=2$ both models belong to the PC class.

The models we define are interesting for various reasons. As generalizations of well-known DP models they give a better physical understanding of models in the PC class. Moreover, they are defined by ordinary two-site nearest-neighbor interactions (rather than three- or four-site interactions) and can be generalized easily to both higher dimensions and a higher number of absorbing states. In addition, unlike previously known models, our models do not explicitly conserve parity. This confirms that the symmetry among the absorbing states is indeed the only essential property of models in the PC class.

Before defining the models let us present an intuitive idea how DP models can be generalized to models with several absorbing states. Directed percolation models are usually defined on some d-dimensional lattice whose sites can be either active (wet) or inactive (dry). If all sites are inactive, the system is in an absorbing configuration from which it cannot escape. In the presence of active sites, the system evolves in time according to specific local processes. Although microscopically these processes may be defined differently, most of the DP models have the feature in common that their time evolution seen on a large scale is subject to the following rules.

(a) Inactive (dry) spots are created randomly within active (wet) islands.

(b) The boundaries between active and inactive domains fluctuate in a way that active islands are biased to grow.

Both processes (a) and (b) compete with each other. Depending on their probabilities, the system can be in two different phases. If the probability for (a) is very small, the system is in the *active phase* where, starting with a nonzero density of active sites, active clusters percolate constantly. If the probability for (a) is very large, the system is in the *inactive phase* where active clusters die out wherefore eventually the system enters the absorbing state. At the percolation threshold the system goes through a continuous phase

transition where the details of the local processes become irrelevant and long-range fluctuations can be observed. According to the DP conjecture [8], we assume that all models with a single absorbing state defined in the spirit of rules (a) and (b) belong to the universality class of directed percolation.

The main idea of the present work is that a generalization of the above rules to n equivalent absorbing states generates universality classes different from DP, in particular to the PC class in the case of two symmetric absorbing states. Such a generalization can be defined as follows. Let us assume that each inactive site carries a ''color'' labeled by $1,\ldots,n$. The simplest generalization of the rules (a) and (b) is the following.

(i) Inactive spots of random color $1,\ldots,n$ are created randomly within active islands.

(ii) The boundaries between active and inactive domains fluctuate in a way that active islands grow.

(iii) Boundaries between inactive domains of different colors are not allowed to stick to each other irreversibly. They are free to separate again leaving active sites in between.

Rules (i) and (ii) are straightforward generalizations of (a) and (b). Again both processes compete with each other and lead to a phase transition from an active to an inactive phase. Rule (iii) is different and distinguishes the different colors. Roughly speaking, this rule tells us that between two inactive domains of different colors a thin film of wet sites is preserved. The importance of this rule becomes obvious by looking at the contrary: If domains of different colors were allowed to stick to each other irreversibly, the colors would then be irrelevant. This would mean that the process is compatible with the previous rules (a) and (b) and thus belong to directed percolation. Rule (ii) allows wet sites between absorbing domains of different colors to survive for a long time. This slows down the relaxation towards one of the absorbing states and therefore we expect systems with several absorbing states to be ''more active'' than usual DP models.

Another important requirement is that the rules are *symmetric* under global permutation of the colors. We will show that if this symmetry is broken, one of the colors begins to play a dominant role so that the phase transition is again in the DP universality class. Although these rules give only an intuitive description rather that a strict definition, they will help us to define models with several absorbing states which will be done in the next section.

II. DEFINITION OF THE MODELS

A. Model I: Generalized Domany-Kinzel cellular automaton

In the Domany-Kinzel model [2] the state at a given time t is specified by binary variables $\{s_i\}$, which can have the values A (active) and I (inactive). At odd (even) times, odd-(even-)indexed sites are updated according to specific conditional probabilities. This defines a cellular automaton with parallel updates (discrete time evolution) acting on two independent triangular sublattices (cf. Fig. 1). The conditional probabilities in the Domany-Kinzel model $P(s_{i,t+1}|s_{i-1,t},\, s_{i+1,t})$ are given by

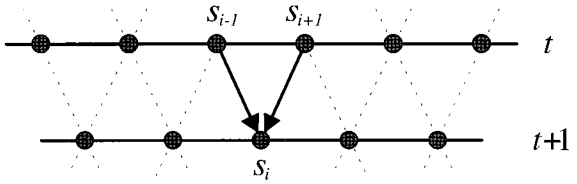

FIG. 1. Update in the Domany-Kinzel model.

$$P(I|I,I)=1, \tag{1}$$

$$P(A|A,A)=q, \tag{2}$$

$$P(A|I,A)=P(A|A,I)=p, \tag{3}$$

and $P(I|s_{i-1},\, s_{i+1})+P(A|s_{i-1},\, s_{i+1})=1$, where $0\leqslant p\leqslant 1$ and $0\leqslant q\leqslant 1$ are two parameters. Equation (1) ensures that the configuration $\ldots,I,I,I,\ldots$ is the absorbing state. The process in Eq. (2) corresponds to rule (a) and describes the creation of inactive (dry) spots within active (wet) islands with probability $1-q$. The random walk of boundaries between active and inactive domains is realized by the processes in Eq. (3). According to rule (b), DP transitions can be observed only if $p>\frac{1}{2}$ when active (wet) islands are biased to grow. The processes and the corresponding probabilities can be summarized in the form of a probability table (cf. Table I).

We now define a generalization of the Domany-Kinzel model following the rules (i)–(iii) (hereafter referred to as model I). This model has $n+1$ states per site: one active state A and n different inactive states $I_1,I_2,\ldots,I_n$. The conditional probabilities are given by $(k,l=1,\ldots,n;\ k\neq l)$

$$P(I_k|I_k,I_k)=1, \tag{4}$$

$$P(A|A,A)=1-nP(I_k|A,A)=q, \tag{5}$$

$$P(A|I_k,A)=P(A|A,I_k)=p_k, \tag{6}$$

$$P(I_k|I_k,A)=P(I_k|A,I_k)=1-p_k, \quad P(A|I_k,I_l)=1, \tag{7}$$

where we study the symmetric case $p_1,\ldots,p_n=p$. Equations (4)–(6) are straightforward generalizations of Eqs. (1)–(3). The only different process is the creation of active sites between two inactive domains of different colors in Eq. (7) according to rule (iii). For simplicity we chose the probability of this process to be equal to one. We may also use a probability less than one, but it turned out that this does not change the critical properties of the system.

For $n=1$ the model defined above reduces to the original Domany-Kinzel model. In Sec. III we will investigate the

TABLE I. Probability table for the ordinary Domany-Kinzel model.

$s_{i-1},\, s_{i+1}$	$P(A\|s_{i-1},\, s_{i+1})$	$P(I\|s_{i-1},\, s_{i+1})$
AA	q	$1-q$
AI	p	$1-p$
IA	p	$1-p$
II	0	1

TABLE II. Probability table for the generalized Domany-Kinzel model with two absorbing states.

s_1,s_2	$P(A\|s_1,s_2)$	$P(I_1\|s_1,s_2)$	$P(I_2\|s_1,s_2)$
AA	q	$1-q/2$	$1-q/2$
AI_1	p	$1-p$	0
AI_2	p	0	$1-p$
I_1A	p	$1-p$	0
I_2A	p	0	$1-p$
I_1I_1	0	1	0
I_1I_2	1	0	0
I_2I_1	1	0	0
I_2I_2	0	0	1

generalized Domany-Kinzel model with $n=2$ absorbing states. The corresponding processes and their probabilities are listed in Table II.

B. Model II: Generalized contact process

The one-dimensional contact process is the simplest example for a DP model with continuous time evolution [3]. Its dynamics is defined by nearest-neighbor processes that occur spontaneously due to specific rates (rather than probabilities). In numerical simulations models of this type are usually realized by random sequential updates. This means that a pair of sites $\{s_i,s_{i+1}\}$ is chosen at random and an update is attempted according to specific transition rates $w(s_{i,t+dt}, s_{i+1,t+dt}|s_{i,t}, s_{i+1,t})$. Each attempt to update a pair of sites (see Fig. 2) increases the time t by $dt=1/N$, where N is the total number of sites. One time step (sweep) therefore consists of N such attempts. The contact process is defined by the rates

$$w(A,I|A,A)=w(I,A|A,A)=\lambda, \tag{8}$$

$$w(I,I|A,I)=w(I,I|I,A)=\mu, \tag{9}$$

$$w(A,A|A,I)=w(A,A|I,A)=1, \tag{10}$$

where $\lambda>0$ and $\mu>0$ are two parameters (all other rates are zero). Equation (8) describes the creation of inactive (dry) spots within active (wet) islands corresponding to rule (a). Equations (9) and (10) describe the shrinkage and growth of active islands according to rule (b). In order to fix the time scale, we chose the rate in Eq. (10) to be equal to one. The active phase is restricted to the region $\mu<1$ where wet islands are likely to grow.

As in the case of the Domany-Kinzel model, we define a generalization of the contact process by introducing n different inactive states $I_1,I_2,\ldots,I_n$. The dynamics of the generalized model (model II) is defined by the rates

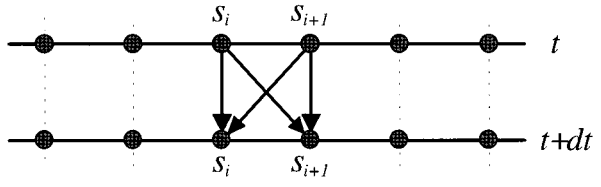

FIG. 2. Update in the contact process.

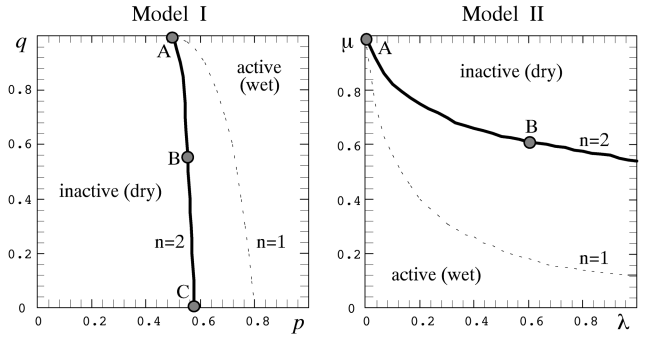

FIG. 3. Phase diagrams of models I and II for two absorbing states. The dashed lines indicate the corresponding transition lines for directed percolation. The explanation of the points A, B, and C can be found in the text.

$$w(A,I_k|A,A)=w(I_k,A|A,A)=\lambda/n, \tag{11}$$

$$w(I_k,I_k|A,I_k)=w(I_k,I_k|I_k,A)=\mu_k, \tag{12}$$

$$w(A,A|A,I_k)=w(A,A|I_k,A)=1, \tag{13}$$

$$w(I_k,A|I_k,I_l)=w(A,I_l|I_k,I_l)=1, \tag{14}$$

where $k,l=1,\ldots,n$ and $k\neq l$ (all other rates are zero). Again we consider the symmetric case $\mu_1,\ldots,\mu_n=\mu$. Equations (11)–(13) are generalizations of Eqs. (8)–(10). Rule (iii) is implemented by Eq. (14), which describes the creation of active sites between two inactive domains of different colors. For $n=1$ the model defined above is reduced to the usual contact process (8)–(10).

C. Phase diagrams

The phase diagrams of both models are shown in Fig. 3. The active (inactive) phase is characterized by a nonzero (vanishing) density of active sites in the thermodynamic limit. Both phases are separated by a phase transition line (the bold line in Fig. 3). The dashed line indicates the corresponding phase transition for a single absorbing state. Comparing both lines we notice that generally models with two absorbing states tend to be more active than their DP counterparts (for exceptions see Ref. [22]).

We checked numerically that as in DP the critical exponents of the generalized models are the same all along the phase transition line. The only exceptions are the ending points $(p,q)=(1/2,\ 1)$ and $(\lambda,\mu)=(0,1)$, where the transition lines for $n=1$ and $n=2$ intersect (marked by A in Fig. 3). In these points rule (i) is no longer valid and the entirely active configuration $\ldots AAA\ldots$ emerges as an additional absorbing state. This leads to a different universality class, which, in the case of $n=1$, is referred to as compact directed percolation.

D. General properties

A typical time evolution of models with two absorbing states is shown in Fig. 4. In the active phase ($\lambda<\lambda_c$) small inactive islands of random color are generated and exist only

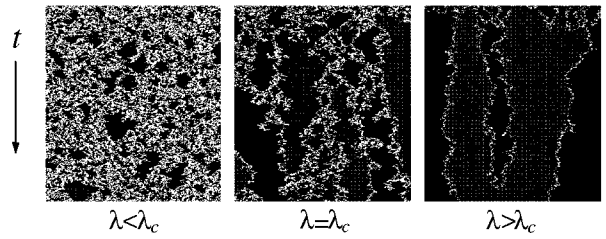

FIG. 4. Simulation of model II for $n=2$ starting from a random initial condition. The two different types of inactive domains are shown in black and gray. The active sites between the domains are represented by white pixels.

briefly. Approaching the phase transition their size and lifetime grows, while the density of active sites decreases. Notice that according to rule (iii) a thin film of active sites separates different inactive domains.

An important property of models with several absorbing states is a very different relaxation towards the absorbing configuration. For DP in the inactive phase the order parameter ρ is known to decay *exponentially* in time. However, this is not true for models with two absorbing states. As shown in Fig. 4 ($\lambda>\lambda_c$), starting from a random initial configuration, large domains of different colors are formed. These domain walls survive and diffuse until they annihilate mutually. In this annihilation process the density of domain walls is known to decay *algebraically* like $\rho(x)\sim t^{-1/2}$ [23]. Because of the slow relaxation numerical simulations of models with several absorbing states are more difficult to perform.

It should be emphasized that in the models defined above there is no explicit parity conservation on the microscopic level: In each local update no more than one site is modified [cf. Eqs. (11)–(14)]. Therefore it is impossible to create more than two kinks or particles per update (a nontrivial parity-conserving dynamics requires the generation of at least three kinks $X\to 3X$, $2X\to 0$). Nevertheless, the annihilating domain walls described above, by their very nature, obey a parity-conserving dynamics. Therefore parity conservation can still be seen on large scales. We will return to this observation in Sec. IV C.

III. TWO SYMMETRIC ABSORBING STATES: NUMERICAL RESULTS

A. Monte Carlo simulations

In order to measure the critical exponents of models I and II in the case of two absorbing states, we perform dynamic Monte Carlo simulations (see, e.g., [19]). We use defect dynamics, i.e., we start with an initial configuration where all sites are in the inactive state I_1 except for one active site in the center. The system then evolves along the dynamic rules of the model. For various values of the parameters near point B in Fig. 3 we perform 10^6 independent runs up to 5000 time steps. However, most of them stop earlier because the system enters into the absorbing configuration where all sites are in the state I_1. (In an infinite system, there is no way to reach the other absorbing configuration I_2). In order to avoid finite-size effects, we adjust the system size after each time step according to the actual size of the active cluster. As usual in this type of simulation, we measure the survival probability $P(t)$, the number of active sites $N(t)$, and the mean square of spreading from the origin $R^2(t)$ averaged over active runs. At criticality, these quantities are expected to scale algebraically in the long-time limit

$$P(t)\sim t^{-\delta}, \quad N(t)\sim t^{\eta}, \quad R^2(t)\sim t^{z}. \tag{15}$$

The critical exponents are related to the exponents β, $\nu_\perp$, and $\nu_{||}$ by

$$\delta=\frac{\beta}{\nu_{||}}, \quad z=\frac{2\nu_\perp}{\nu_{||}} \tag{16}$$

and obey the scaling relation

$$4\delta+2\eta=dz. \tag{17}$$

The quantities (15) show straight lines in double logarithmic plots. Off criticality, the lines are curved. In order to get precise estimates for the scaling exponents, it has been useful to consider the local slopes of the curves by introducing the *effective exponents*

$$-\delta(t)=\frac{\log_{10}[P(t)/P(t/b)]}{\log_{10}b} \tag{18}$$

and similarly $\eta(t)$ and $z(t)$, where $\log_{10}b$ is the distance used for estimating the slope. Choosing $b=5$, we measured the effective exponents of both models for various values of $q=p$ and $\lambda=\nu$. The results of our simulations are shown in Figs. 5 and 6. Off criticality, the curves for $\delta(t)$ and $\eta(t)$ show negative or positive curvature. The figures give us an estimate of the critical points $p_c=0.5673(5)$ for model I and $\lambda_c=0.628(2)$ for model II. The estimates for the critical exponents are $\delta=0.285(10)$, $\eta=0.00(1)$, and $z=1.15(1)$ for model I and $\delta=0.29(1)$, $\eta=0.00(1)$, and $z=1.15(1)$ for model II.

The exponent β has been obtained directly in static simulations by measuring the steady-state density ρ in the active phase near the critical point. Although this method is known to be quite inaccurate to determine β and we measured only over one decade in ϵ, we found the reasonable values $\beta=0.90(5)$ for model I and $\beta=0.93(5)$ for model II.

The estimates of the critical exponents agree with previous results for models in the PC class (cf. Table III). Thus, from our numerical results we can conclude that for $n=2$ both models belong to the PC universality class.

B. Symmetry-breaking field

Recently Park and Park showed in the example of an interacting monomer-dimer model that if the symmetry among the absorbing states is broken by an external field, the DP universality class is recovered [21]. In order to verify this observation, we introduce an external field by modifying the growth rates for inactive islands of different colors. This can be done by choosing different p_k in Eq. (6) [μ_k in Eq. (12)]. Because of the different growth rates, one of the colors is going to play a dominant role wherefore in the large-scale limit the system evolves as if it had only a single absorbing state.

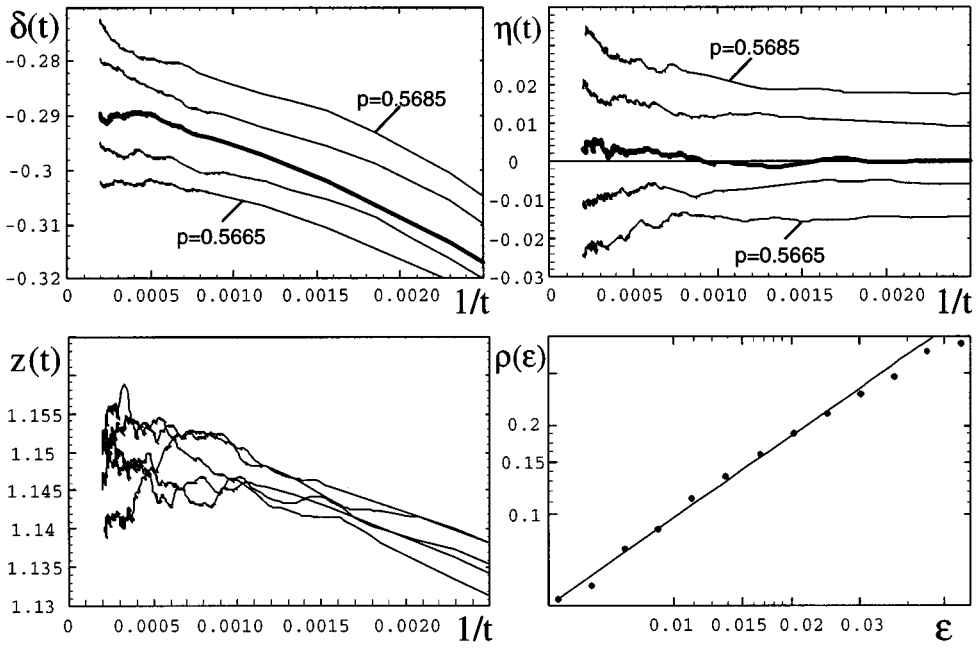

FIG. 5. Numerical results for model I. The effective exponents $\delta(t)$, $\eta(t)$, and $z(t)$ are obtained from dynamic simulations for $p=q=0.5665,\ 0.5670, \ldots, 0.5685$. The densitiy of active sites $\rho(\epsilon)$ is measured in static simulations for different values of $\epsilon = p - p_c$. The slope of the line in the log-log plot gives an estimate for the exponent β.

To demonstrate this, we repeat the above simulations of model I for $p_{1,2}=q\pm 0.02$. The critical point is shifted to $q_c=0.6245(10)$. The exponents we measure (cf. Fig. 7) are $\delta=0.16(2)$, $\eta=0.32(4)$, $z=1.27(4)$, and $\beta=0.28(1)$, which agree with the DP exponents [18] $\delta=0.159(1)$, $\eta=0.314(1)$, $z=1.266(2)$, and $\beta=0.2765(1)$.

If the symmetry of a model with $n>2$ absorbing states is partially broken, a subset of m colors starts to play a dominant role. We expect that such a system behaves at criticality like a model with m absorbing states.

IV. OTHER SPECIAL CASES

A. Compact clusters

As mentioned before, the Domany-Kinzel model and the contact process have a line in their phase diagram where the entirely active configuration emerges as an additional absorbing state. On this line active clusters are not fractal but compact, wherefore it is called the compact directed percolation line. Here the dynamical processes are exactly solvable by reducing them to an annihilation-diffusion process of kinks $2X\rightarrow 0$. Although this leads to a different universality class at the phase transition point (point A in Fig. 3), compact DP has been used in many cases to improve the understanding of ordinary DP.

A similar situation exists in generalized models with several absorbing states. The fact that the transition A point is identical to that of ordinary compact DP indicates that this point is exactly solvable in all cases. However, the translation into a kink language is slightly more complicated. Since kinks between inactive domains of different colors cannot exist [rule (iii)], only n types of kinks X_k between active and

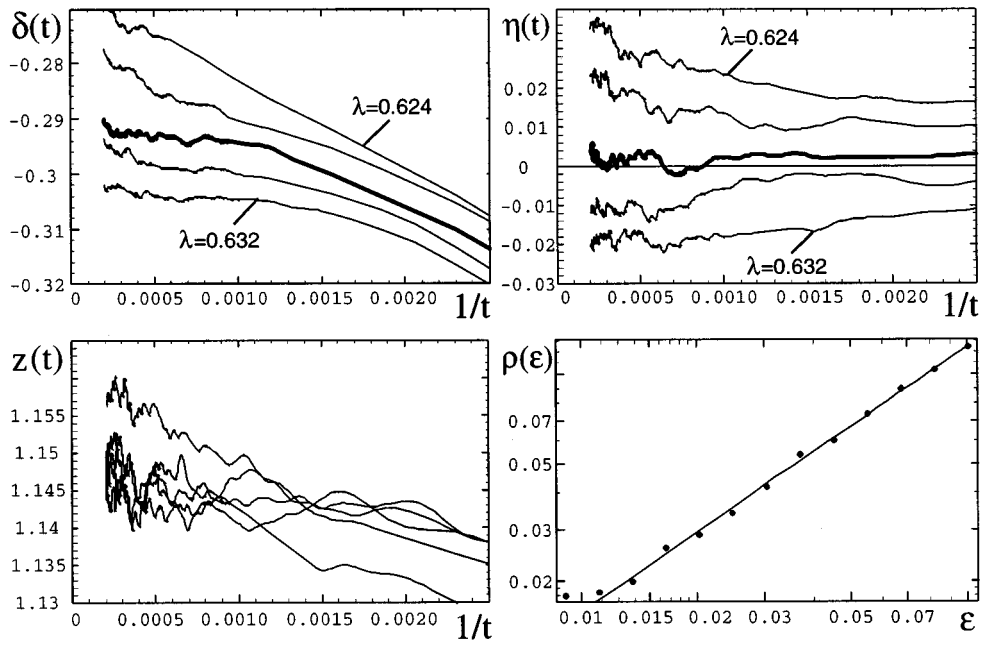

FIG. 6. Results of analog simulations of model II. The parameters vary in the range $\lambda=\mu=0.624,\ 0.626, \ldots, 0.632$.

TABLE III. Critical exponents of models in the PC class. The asterisks indicate values measured in kink dynamics where a different scaling relation holds.

Model	δ	η	z	β
	Known models:			
A and B [12]	0.27(8)			0.6(2)
BAW $n=2$ [13]	0.283(16)	0.272(12)*	1.11(2)	0.94(6)
BAW $n=2$ [19]	0.285(2)	0.000(1)	1.141(2)	0.92(3)
BAW $n=4$ [18]	0.286(2)	0.000(1)	1.147(4)	0.922(5)
kinetic Ising [14]	0.27(2)	0.30(2)*	1.14(2)	0.80(8)
dynamic BAW [22]	0.287(1)	0.000(3)	1.155(5)	
MDM [15]	0.29(2)	0.00(2)	1.34(20)	0.88(3)
	Present work			
I	0.285(10)	0.00(1)	1.15(1)	0.90(5)
II	0.29(1)	0.00(1)	1.15(1)	0.93(5)

inactive domains (AI_k, I_kA) play a role. These kinks have to occur in pairs and undergo an annihilation-diffusion process $2X_k \to 0$ (kinks of different types cannot annihilate). It is important that there is no more generation of randomly colored inactive domains. Therefore dynamical simulations such as those in Sec. III, starting from an initial configuration with only one type of inactive sites, yield the same results for all n, i.e., $\delta = 1/2$ and $z = 1$. However, compact percolation processes are known to depend strongly on initial conditions so that generally the situation may be more complicated.

B. Simulations in higher dimensions

The models presented in this paper can easily be generalized to higher dimensions. This is particularly simple in the case of model II since its definition (11)–(14) can be used in any dimension. In simulations we observed that for $d=2$ this system has a phase transition, although the relaxation towards one of the absorbing states in the inactive phase is extremely slow. Figure 8 shows typical configurations in the active phase near criticality. While in ordinary DP active clusters are separated spatially, active sites in two-dimensional models with two absorbing states are arranged in fractal ''lines'' along the boundaries of inactive islands. Repeating the simulations described above on a 80×80 lattice we obtained $\lambda_c = 0.99(1)$, $\delta = 0.9(1)$, $\eta = 0.00(5)$, and $z = 1.0(1)$. These results agree roughly with the mean-field exponents $\delta=1$, $\eta=0$, $z=1$ and $\beta=1$ (see Sec. IV C). Therefore, we conjecture that $1 < d_c \leqslant 2$ is the upper critical dimension of systems with two absorbing states.

C. Relation to BAW's

We already mentioned that models I and II have the same critical behavior all along the phase transition line (except point A in Fig. 3). Moving along this line we can control the mean size of active islands, which is infinite at point A, a few sites at point B, and one site at point C (in model II the latter case corresponds to taking $\lambda \to \infty$). In one dimension point C can be related to branching walk models since each active site can be interpreted as a single walker. For $n=1$ these walkers diffuse and interact by $2X \leftrightarrow X$, $X \to 0$, which results in a DP process. For $n \geqslant 2$ the situation is more complicated. Let us denote by X_{jk} a walker separating inactive domains I_j and I_k. Then the processes are $X_{jk} \leftrightarrow X_{jl}X_{lk}$ (with random l) and $X_{jj} \to 0$. These processes clearly do not conserve parity. However, because of $X_{jj} \to 0$ only walkers between domains of different color $j \neq k$ survive for a long time [cf. rule (iii)]. Hence, for $n=2$ the majority of walkers react like

$$X_{12} \to X_{11}X_{12} \to X_{12},$$

$$X_{12} \to X_{11}X_{12} \to X_{12}X_{21}X_{12},$$

$$X_{12}X_{21} \to 0$$

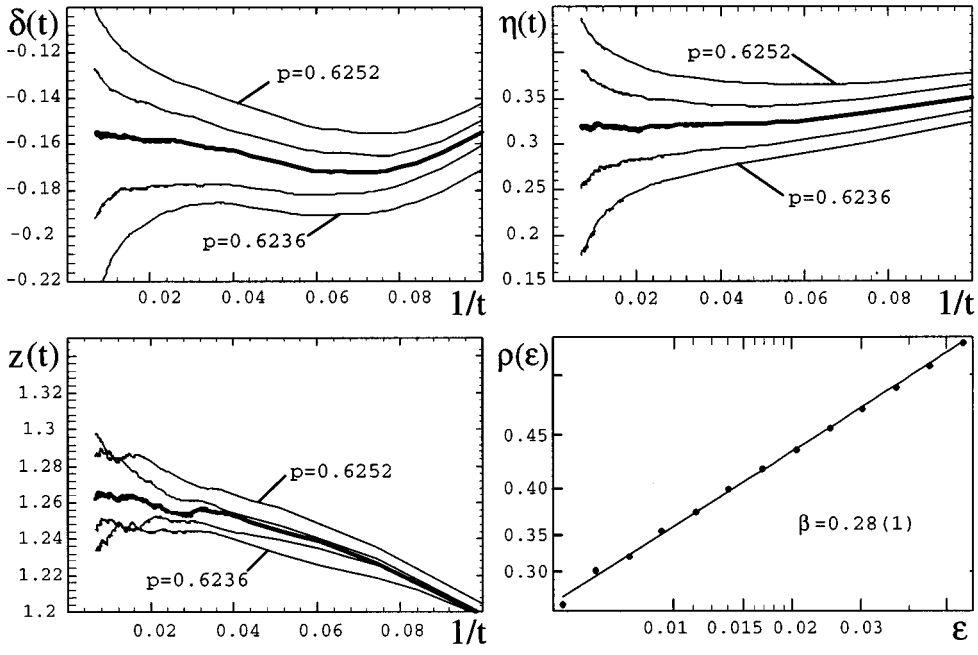

FIG. 7. Symmetry-breaking field: simulation of model I for slightly different growth rates ($p_{1,2} = q \pm 0.02$). DP exponents are recovered.

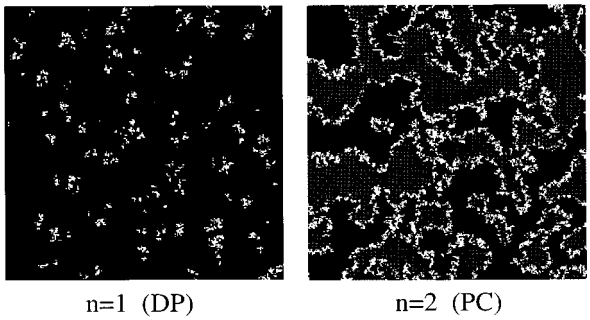

FIG. 8. Simulation snapshot of two-dimensional systems with one and two absorbing states in the active phase near criticality. White dots denote active sites.

(with analog reactions for X_{21}). Thus, in a long-time limit the walkers undergo an effective reaction of the type $X\to 3X$, $2X\to 0$, which is a BAW with two offspring in one dimension.

The relation to BAW modes is even more general and holds not only at point C but everywhere on the phase transition line except for point A. The walkers then have to be identified with domain walls separating inactive domains of different colors. At first glance this seems to be contradicting: Domain walls, by their very definition, obey a local parity-conserving dynamics. On the other hand, it is obvious that the local processes do not conserve parity. However, domain walls in our models are extended objects. Their thickness fluctuates and varies along the phase transition line from typically one site at point C up to infinitely many sites at point A. It is important to notice that the domain walls simply cannot be identified with active islands (which may also occur between inactive domains of the same color) but require a more complicated definition. Although there is no microscopic parity-conservation, a careful analysis shows that the dynamical rules ensure that all microscopic processes violating parity conservation have a very short lifetime. This is the reason why an effective parity-conserved dynamics can be recovered in the limit of large scales in time and space.

In higher dimensions $d\geqslant 2$ the physical properties of BAW's with an even number of offspring are governed by the mutual annihilation of the walkers [24]. However, the models presented in this paper behave very differently in higher dimensions, which makes it impossible to relate them to BAW's. As shown in Fig. 8, the active sites in models I and II arrange themselves as $(d-1)$-dimensional surfaces separating inactive domains of different colors. Thus they cannot be interpreted as pointlike random walkers. In $d\geqslant 2$ dimensions we therefore expect BAW's with an even number of offspring to be in a different universality class from the present models.

D. More than two absorbing states

No results were obtained for $n\geqslant 3$ symmetric absorbing states in one dimension. It turned out that it is impossible to determine the critical point because the plots for $\delta(t)$, $\eta(t)$, and $z(t)$ show only one type of curvature. This observation agrees with recent results obtained for an N-species generalization of BAW models with an even number of offspring [20]. For $N\geqslant 2$ these models are always in the active phase and their critical exponents are described by yet another universality class.

E. Mean-field approximation

Denoting by ρ_k the density of inactive sites I_k, the processes in model II imply the mean-field equation

$$\frac{\partial}{\partial t}\rho_k=\frac{2\lambda}{n}\rho_0^2+\mu\rho_0\rho_k+\rho_k^2-\rho_k, \tag{19}$$

where $\rho_0=1-\Sigma_{j=1}^{n}\rho_j$. Choosing $\lambda=\mu$, the critical point is $\lambda_c=1$ for $n=1$ and $\lambda_c=\infty$ for $n\geqslant 2$, which means that mean-field models with more than two absorbing states are always in the active phase (see also Ref. [20]). For all n the density of active sites ρ_0 scales like $(\lambda-\lambda_c)^{\beta}$ with the mean-field exponent $\beta=1$. This is close to the measured values $\beta\sim 0.9$ in the model with two absorbing states, which indicates that we are already close to the upper critical dimension.

V. SUMMARY AND DISCUSSION

We have shown by the example of the Domany-Kinzel model and the contact process that lattice models for directed percolation can be generalized to models with n symmetric absorbing states. Numerical simulations lead to the conclusion that such models in one dimension with two absorbing states belong to the PC universality class. Since these models do not explicitly conserve any quantity mod2 they show that rather than parity conservation, the symmetry among the absorbing states is the origin for the emergence of a different class. As soon as this symmetry is broken, the critical behavior jumps back to DP.

The symmetry used in our models is the group of permutations S_n. No reliable numerical results could be obtained for S_3 and higher symmetries. However, one may also introduce other symmetries such as cyclic groups (e.g., Z_3) and investigate whether they define different universality classes (cyclic symmetries of this type appear, e.g., in the three-candidate voter model [25]). It would be also interesting to examine models with more than one symmetric active state.

For a better understanding it would be desirable to find an appropriate field-theoretical description of the model. For BAW's with an even number of offspring this has been done recently in Ref. [20]. There are surprising results; in particular one has two different critical dimensions: one of them ($d_{c_1}=4/3$) related to the properties of active clusters and the other ($d_{c_2}=2$) related to the annihilation process. Although the application of this theory to the present type of model may not be transparent, Cardy and Täuber were able to identify an S_2 symmetry on an operator level. This again indicates that this symmetry plays an important role in the PC class.

A field theory for the present type of model should be different from that of BAW's because of its different phenomenology in higher dimensions (see Secs IV B and IV C). Nevertheless, both theories should give identical results in one dimension. The simplest ansatz for such a field theory

would be to add a diffusionlike term and n noise fields to the mean-field equations

$$\frac{\partial}{\partial t}\rho_k(x,t)=\frac{2\lambda}{n}\rho_0^2(x,t)+\mu\rho_0(x,t)\rho_k(x,t)+\rho_k^2(x,t) -\rho_k(x,t)+D\nabla^2\rho_k(x,t)+\eta_k(x,t). \quad (20)$$

However, even if this were correct, the derivation of the correlations in the noise remains a highly nontrivial problem. Therefore, the development of appropriate field theories is a challenge towards a better understanding of universality classes appearing in systems with several absorbing states.

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